Mathematics (AHS)

January 2016

Part 2 Tasks

Bundesinstitut bifie
Bildungsforschung, Innovation & Entwicklung
des österreichischen Schulwesens

Task 1

Quadratic Equations and their Solutions

You will be working with a (monic) quadratic equation of the form $x^2 + p \cdot x + q = 0$, where $p, q \in \mathbb{R}$ and its corresponding polynomial function, f, which has equation $f(x) = x^2 + p \cdot x + q$.

Task:

a) If the equation $x^2 + p \cdot x + q = 0$ can be written in the form $(x - z) \cdot \left(x - \frac{1}{z}\right) = 0$, where $z \in \mathbb{R}$ and $z \neq 0$, it is known as a reciprocal quadratic equation.

   By means of equations, show how the parameters p and q each depend on z.

   Determine the values of z such that the reciprocal quadratic equation has exactly one solution. For each of these values, find the local minima of the function f.

b) If you substitute the value $q = -1$ into the original function, you have a second degree polynomial function f, where $f(x) = x^2 + p \cdot x - 1$.

   $\boxed{\text{A}}$* Show by means of calculation that the equation $f(x) = 0$ has exactly two distinct solutions in $\mathbb{R}$.

   Explain how you know that the equation f has one positive and one negative root.

c) For the case when $q = p - \frac{1}{3}$, the function f has equation $f(x) = x^2 + p \cdot x + p - \frac{1}{3}$.

   Determine the value of p such that $\int_{-1}^{1} f(x)\,\mathrm{d}x = -6$.

   For this value of p, does the equation $\int_{-1}^{0} f(x)\,\mathrm{d}x = \int_{0}^{1} f(x)\,\mathrm{d}x$ hold? Justify your answer.

* The tasks marked with an $\boxed{\text{A}}$ may serve as compensation points in the written examination and may be used to compensate for the shortfall (as part of the "range of essential skills" outlined by the LVBO).

Task 2

Design-Center Linz

The Design-Center is one of the modern landmarks of the city of Linz. It was built between July 1991 and the end of October 1993. In January 1994, it was opened as a business and exhibition centre.
Parabolic arcs can be used as a good approximation of the shape of the supports of the structure. The span of the arcs is approximately 72 m, and the maximum height of the arcs is about 13 m. The footprint of the Design-Center is a rectangle with a length of 200 m and a width of 72 m.

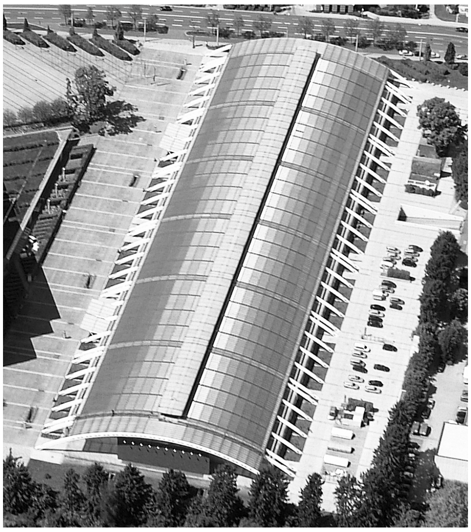
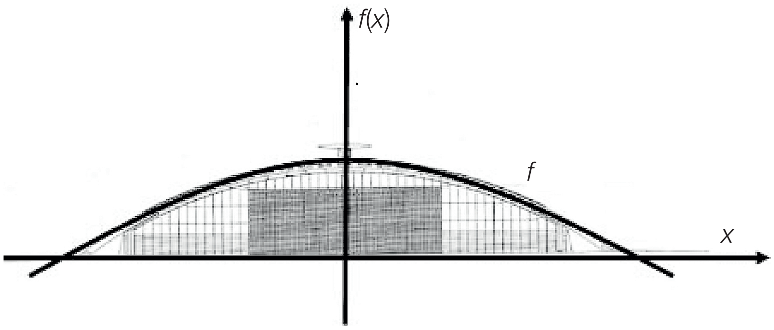


Image source: http://www.linz.at/images/dc_druck.jpg [09.09.2015]

Task:

a) In order to model the parabolic supports, a coordinate system is placed over a diagram of the front elevation of the Design-Center, as shown below:



$\boxed{A}$ Write down a second degree polynomial function that describes this parabola.

What can be calculated from $200 \cdot 2 \cdot \int_0^{36} f(x)\,\mathrm{d}x$ in the given context?

b) The construction costs for the Design-Center at the time of completion (1993) came to the equivalent of around € 66 million.
The Construction Cost Index is a measure of how much costs that construction companies incur from building works change over time. It is based on the trends of the price of the basic costs involved in construction (materials and labour). It indicates, for example, how dramatically the costs for construction projects increase each year. Given that the Construction Cost Index for Austria is 3.5 % per year, calculate how high the construction costs of the Design-Center would have been if it had been built 10 years later.

The table below gives an indication of the trend of the Construction Cost Index of complete construction costs for housing and residential construction over a period of five consecutive years.

Year	Construction Cost Index
2010	+3.2 %
2011	+2.3 %
2012	+2.1 %
2013	+1.9 %
2014	+1.1 %

Source: http://www.statistik.at/web_de/statistiken/wirtschaft/preise/baukostenindex/index.html [30.10.2015]

A man planning to build his own house is interested in finding out the average Construction Cost Index over these five years. He attempts to estimate this value by carrying out the following calculation:

$$\frac{3.2 + 2.3 + 2.1 + 1.9 + 1.1}{5} = 2.12$$

However, his method is not entirely appropriate for the calculation of the average Construction Cost Index. Show how this calculation should have been done correctly.

Task 3

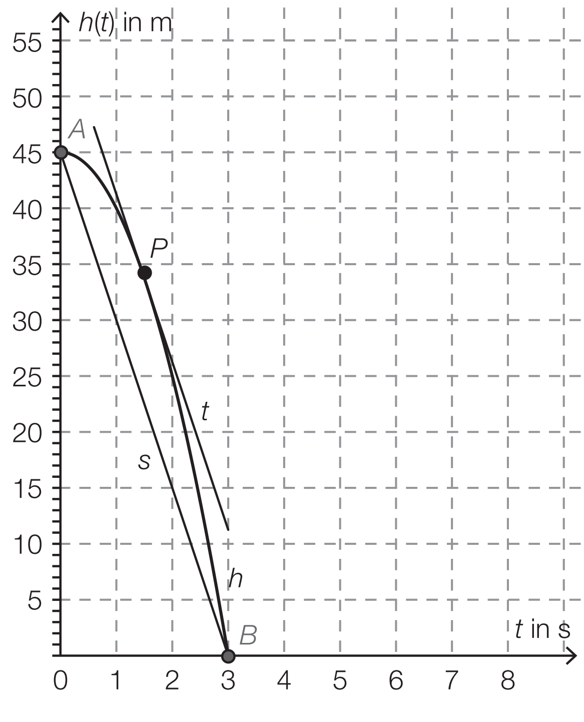
The Leaning Tower of Pisa

The Leaning Tower of Pisa is one of the most famous buildings in the world.
Galileo Galilei's (1564–1642) studies of falling objects from different heights of the Leaning Tower of Pisa are not historically documented. However, it is known that Galilei researched the laws of motion under gravity. If air-resistance is ignored (i. e. if the experiment is conducted in a vacuum), the time taken for a body to fall from a height h_0 is independent of its shape and its mass.
The height of a falling body with respect to time can be approximated by the function h, where $h(t) = h_0 - 5t^2$. The height $h(t)$ is measured in metres and the time t is measured in seconds.

Task:

a) A body falls in a vacuum from a height $h_0 = 45$ m.
$\boxed{\text{A}}$ Determine its velocity in m/s at time t_1, the moment of impact with the floor.
Explain why the magnitude of the velocity of the body in the interval $[0, t_1]$ increases monotonically.

b) In the diagram below, the graph of the function h for $h_0 = 45$ m is shown.
Determine the gradient of the secant s, which goes through the points $A = (0,45)$ and $B = (3,0)$. Explain the significance of this result with respect to the movement of the body.



The tangent t at point $P = (1.5, h(1.5))$ is parallel to the secant s. Interpret this result with respect to the movement of the body.

Task 4

Reaction Test

In a computer-based reaction test, the participants were shown 20 patterns on the screen consecutively that had to be classified. The total time taken to complete the whole reaction test, t, as well as the number of incorrect classifications, f, were recorded.
The table below shows the results of a series of ten tests (where each test required 20 reactions as described above) for one participant.

Test number	t (in seconds)	f
1*	$t_1 = 22.3$	$f_1 = 3$
2	$t_2 = 24.6$	$f_2 = 2$
3	$t_3 = 21.8$	$f_3 = 3$
4	$t_4 = 23.5$	$f_4 = 1$
5	$t_5 = 32.8$	$f_5 = 5$
6	$t_6 = 21.7$	$f_6 = 4$
7	$t_7 = 22.6$	$f_7 = 3$
8	$t_8 = 22.8$	$f_8 = 2$
9	$t_9 = 35.4$	$f_9 = 3$
10	$t_{10} = 22.5$	$f_{10} = 1$

* Explanation: For the first test the participant took 22.3 seconds and three of his/her classifications were incorrect.

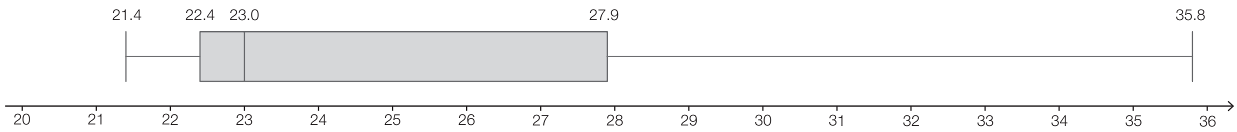
Task:

a) $\boxed{A}$ Calculate the arithmetic mean, $\overline{t}$, of the ten reaction times $t_1, t_2, \dots, t_{10}$ as well as the standard deviation, s_t, of these ten values.

The participant took the reaction test two more times and completed it in the times t_{11} and t_{12}. The arithmetic mean of the new data set $t_1, t_2, \dots, t_{10}, t_{11}, t_{12}$ is denoted by $\overline{t}_{\text{new}}$, and the corresponding standard deviation is denoted by s_{new}. Identify values for t_{11} and t_{12} such that $t_{11} \neq t_{12}$, $\overline{t}_{\text{new}} = \overline{t}$ and $s_{\text{new}} < s_t$.

b) During a discussion of the results, someone is of the opinion that the arithmetic mean of the 10 reaction test times does not describe the data particularly well. Suggest a possible argument that supports this opinion as well as an alternative statistical tool.

The data set of the 500 reaction times of 50 participants is represented by the following box plot.



Is the following statement correct: “At most 125 of the 500 reaction test times were at most 22.4 seconds”? Justify your answer.

c) The random variable *H* maps each test attempt, in which a participant is shown 20 pictures, to the number of incorrect reactions made by the participant.
On the basis of this information, describe the conditions the reaction test should fulfil such that the random variable *H* can be described by a binomial distribution.

Calculate $P(H > 2)$ if a participant reacts incorrectly with a probability of $p = 0.15$.

Task 5

Surprise Eggs

An Italian confectionary manufacturer produces the product known as the *Kinder Surprise Egg* (also known als “Surprise Egg”). The egg is made of 20 g of chocolate. Inside the egg there is a toy inside a yellow capsule. The shape of this capsule can be approximated by a cylinder that has a hemisphere attached to its base and its top. The volume of the capsule is approximately 36 cm^3 and its surface area is 55 cm^2.

Image source: https://de.wikipedia.org/wiki/Datei:Überraschungsei.jpg [01.06.2015]
(author: A. Kniesel, licence: CC BY-SA 3.0)

Task:

a) During the quality control process, eggs whose mass deviates from the prescribed value by more than 0.5 g are discarded. During an inspection, 500 chocolate eggs were randomly chosen from a production line and checked. 15 of these eggs were discarded.

State a symmetrical 90 % confidence interval for the relative proportion, p, of discarded eggs for the whole production line.

Describe which measures could be undertaken to reduce the width of the current confidence interval at the given confidence level.

b) The manufacturer is considering producing the yellow capsules only in the shape of a cylinder without the hemispheres on the top and base in the future. The volume, V, of the capsule should remain the same, as should the shape of the chocolate egg. The surface area, $O(r)$, of this cylinder can be written as a function of its radius, r, in the equation $O(r) = 2r^2\pi + 2 \cdot V \cdot r^{-1}$. The radius r can only take values in the interval (0 cm, 1.9 cm] so that the cylinder still fits inside the chocolate egg.

Calculate the smallest possible surface area that the new cylinder could have.

Demonstrate by means of differentiation that the value found is indeed a minimum point.